Applications of Extrema

Motivating Example : What is the minimum value the sum of two positive numbers can be if their product is 100 ?

Quantity unknowns : $x, y \geq 0$

Identify objective quantity to be minimized / maximized : $x+y$

Identify constraint : $xy = 100$

Solve constraint in one unknown and substitute into objective :

$$xy = 100 \Rightarrow y = \frac{100}{x}$$

$$\Rightarrow x+y = x + \frac{100}{x} = f(x)$$

Identify domain and find absolute extrema with appropriate technique :

$$x, y \geq 0 \text{ and } xy = 100 \Rightarrow x > 0$$

Need to find absolute min of $f(x) = x + \frac{100}{x}$ on $(0, \infty)$

$(0,\infty) \neq$ closed interval $\Rightarrow$ Must use critical point theorem/ global sign analysis of $f'(x)$.

$$f'(x) = 1 - \frac{100}{x^2}$$

A/ $f'(x) = 0 \Rightarrow 1 - \frac{100}{x^2} = 0 \Rightarrow x^2 = 100 \Rightarrow x = \pm 10$

B/ f' undefined when $x = 0$

$f(10)$ exists

0    $-$ 10 $+$ $f'(x)$

$f'(1) = -99 < 0$ $f'(100) = 1 - \frac{1}{100} = \frac{99}{100} > 0$

$\Rightarrow f(10) = 10 + \frac{100}{10} = 20$ is an absolute minimum of $f(x)$ on $(0,\infty)$. $\left(x = 10, \; y = \frac{100}{x} \Rightarrow y = 10 \right)$

Gather it all together into single conclusion :

If $x, y \geq 0$ and $xy = 100$ then the minimum value of $x+y$ is 20 when $x = 10$ and $y = 10$.

Remark : A problem like this is called a constrained optimization problem.

Strategy to Solve Constrained Optimization Problems

1/ Read problem very carefully. Identify what objective quantity you are being asked to maximize / minimize.

2/ Perhaps draw a picture and label unknowns (generally there will be two degrees of freedom in this course)

3/ Express objective quantity in terms of unknowns.

4/ Identify what constraint is imposed by problem. Express as constraint equation in unknowns.

5/ Solve constraint equation in one variable and substitute into objective to get a single variable function f.

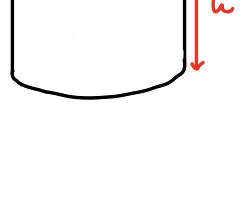
6/ Identify domain imposed by f and nature of problem and find absolute max/min.

7/ Conclude by explicitly stating the solution to the problem.

Example A coffee company wants to manufacture cylindrical coffee cans with a volume of 1000 cm^3. What should the radius and height be to minimize surface area.

1/ We are trying to minimize surface area of a cylinder

2/



3/ Surface area = Area of top and bottom

+ Area of sides

$$= \pi r^2 + \pi r^2 + \underbrace{2\pi r}_{\text{circumference}} \times h = \underbrace{2\pi r^2 + 2\pi r h}_{\text{Objective}}$$

4/ Constraint : Volume is 1000

Volume = (area of base) × height

$= \pi r^2 . h$

$\Rightarrow \pi r^2 h = 1000$

5/ Solve constraint in h : $\pi r^2 h = 1000$

$\Rightarrow h = \frac{1000}{\pi r^2}$

Substitute into objective :

$$2\pi r^2 + 2\pi h = 2\pi r^2 + 2\pi r \cdot \frac{1000}{\pi r^2} = 2\pi r^2 + \frac{2000}{r} = f(r)$$

6/ $r, h \geqslant 0$ and $\pi r^2 h = 1000 \Rightarrow r > 0$

Need to minimize $f(r) = 2\pi r^2 + \frac{2000}{r}$ on $(0, \infty)$

$f'(r) = 4\pi r - \frac{2000}{r^2}$

A/ $f'(r) = 0 \Rightarrow 4\pi r - \frac{2000}{r^2} = 0 \Rightarrow 4\pi r = \frac{2000}{r^2}$

$\Rightarrow r^3 = \frac{500}{\pi} \Rightarrow r = \left(\frac{500}{\pi}\right)^{\frac{1}{3}} \approx 5.419$

B/ f' undefined $\Rightarrow r = 0$

C.N.

0 — $\left(\frac{500}{\pi}\right)^{1/3}$ + $f'(r)$

$f'(1) = 4\pi - 2000 < 0$ $f'(100) = 400\pi - \frac{2000}{10000} > 0$

$\Rightarrow f\left(\left(\frac{500}{\pi}\right)^{1/3}\right)$ absolute min of f on $(0, \infty)$

$r = \left(\frac{500}{\pi}\right)^{1/3}$, $h = \frac{1000}{\pi r^2}$ $\Rightarrow$ $h = \frac{1000}{\pi\left(\frac{500}{\pi}\right)^{2/3}}$

7/ The surface are is at a minimum when

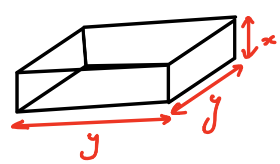
$r = \left(\frac{500}{\pi}\right)^{1/3}$ and $h = \frac{1000}{\pi\left(\frac{500}{\pi}\right)^{2/3}}$

<u>Example</u> A supplier wants to create an open box from a 12 inch by 12 inch square of metal by cutting squares from the corners and folding up the sides.

What is the maximum volume ?

1/ Maximizing volume of a rectangular box with square base.

2/



3/ Volume $= y^2 x$

4/ Constraint : $y + 2x = 12$, $x, y \geq 0$

5/ $y + 2x = 12 \Rightarrow y = 12 - 2x$

$\Rightarrow y^2 x = (12-2x)^2 x = f(x)$

f is continuous on closed interval.

6/ $x, y \geq 0$ and $y + 2x = 12 \Rightarrow x$ in $[0, 6]$

$f(x) = 144x - 48x^2 + 4x^3$

$\Rightarrow f'(x) = 144 - 96x + 12x^2 = 12(x^2 - 8x + 12) = 0$

$= 12(x-6)(x-2)$

A/ $f'(x) = 0 \Rightarrow x = 6$ or 2

B/ f' continuous everywhere.

$\Rightarrow$ $0, 2, 6$ are all critical number of $f(x)$ on $[0,6]$

$$\left.\begin{array}{l} f(0) = 0 \\ f(2) = 128 \\ f(6) = 0 \end{array}\right\} \Rightarrow f(2) = 128 \text{ is absolute max on } [0,6]$$

7/ The boxes maximum volume is $128\ in^3$. This occurs when it has height 2 in and width 8 in.

<u>Remark</u> Some times one might quantify the problem with one unknown variable at the very start. Generally speaking this is just blending together 3/, 4/ and 5/. I'd stick with the approach outlined above. See example 3 on P333 of textbook.